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1. Flow on non-Newtonian fluids of the rate type between two parallel plates rotating with different angular speeds about a common axis.
2. Flow of non-Newtonian fluids of the integral type between parallel plates rotating about distinct axes.
3. Flow of non-Newtonian fluids due to torsional and longitudinal oscillations.
4. Flow of non-Newtonian fluids in pipes of varying cross-sections.
5. Flow of shear thinning fluid between intersecting planes.

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FINAL REPORT: **INVESTIGATIONS INTO SWIRLING FLOWS OF NEWTONIAN AND NON-NEWTONIAN FLUIDS**

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Papers published which were supported by the AFOSR Grant

1) Multiplicity of solutions in von Karman flows of viscoelastic fluids, *J. of Non-Newtonian Fluid Mechanics*, **36**, 1- (1990)(Z. Ji, K.R. Rajagopal and A.Z. Szeri).

2) A numerical study of the flow of a K-BKZ fluid between plates rotating about non-coincident axes, *J. of Non-Newtonian Fluid Mechanics*, **38**, 289 - (1991)(R.X. Dai, K.R. Rajagopal and A.Z. Szeri).

3) Flow of viscoelastic fluids between rotating disks, *Theoretical and Computational Fluid Dynamics*, **3**, 185- (1992) (K.R. Rajagopal).

4) Flow of a non-Newtonian fluid through axi-symmetric pipes of varying cross-sections, In press, *International J. of Non-Linear Mechanics* (S. Kasivishvanathan, P.N. Kaloni and K.R. Rajagopal).

5) Flow of an incompressible simple fluid due to longitudinal and torsional oscillations of a cylinder, *J. of Mathematical and Physical Sciences*, **23**;, 445 (1989) (K.R. Rajagopal, P.N. Kaloni and L. Tao).

6) Some recent results on swirling flows of Newtonian and non-Newtonian fluid, In Proceedings of III International Workshop on Mathematical Aspects of Fluid and Plasma Dynamics, Salice-Terme, Italy (1989) (Will appear as a part of Lecture Notes in Mathematics by Springer-Verlag) (K.R. Rajagopal).

7) Flow of a shear-thinning fluid between intersecting planes, *International J. of Non-Linear Mechanics*, **26**, 769 (1991) (D. Mansutti and K.R. Rajagopal).

The grant also supported the doctoral dissertation of M. Z.H. Ji titled "Multiplicity of solutions of von Karman flows" awarded in 1990.

INTRODUCTION

As we mentioned in the previous progress report, in addition to the studies on swirling flows of Newtonian and non-Newtonian fluids, we also studied other closely related problems involving flows of non-Newtonian fluids which can shed light on the mechanics of such fluids. In all, we studied the following problems:

(i) Flow of non-Newtonian fluids of the rate type between two parallel plates rotating with different angular speeds about a common axis.

(ii) Flow of non-Newtonian fluids of the integral type between parallel plates rotating about distinct axes.

(iii) Flow of non-Newtonian fluids due to torsional and longitudinal oscillations.

(iv) Flow of non-Newtonian fluids in pipes of varying cross-sections.

(v) Flow of shear thinning fluid between intersecting planes.

The results of our work have been published in archival journals and copies of these have been attached to this final report. Here, we discuss briefly our contributions to the understanding of each of the above problems. It would be fair to say that this study has contributed significantly to the understanding of swirling flows in both Newtonian and non-Newtonian fluids and this is reflected in the various invitations we have received both nationally and internationally to present our work.

The grant also supported the doctoral dissertation of Mr. Z.H. Ji which was titled "Multiplicity of von Karman flows of viscoelastic fluids" awarded in 1990.

MATHEMATICAL PRELIMINARIES

Let $\Omega_0 \subset E$ denote the reference configuration of the body. We shall denote by $\mathbf{X} \in \Omega_0$, particles belonging to the body in the reference configuration. Let the

configuration of the body at time t be denoted by Ω_t and let $\mathbf{x}$ denote the position of $\mathbf{X}$ at time t. By the motion of the body, we mean a one-to-one invertible mapping $\mathbf{x}$ through

$$x = \chi(X,t). \tag{1}$$

The deformation gradient $\mathbf{F}$ associated with the motion is defined through

$$F = \frac{\partial \chi}{\partial X}. \tag{2}$$

We shall assume that $\mathbf{F}$ is invertible. Let ξ denote the position of the particle at time τ, $0 \le \tau \le t$. Then

$$\xi = \chi(x,\tau) = \chi\left(\chi^{-1}(x,t),t\right) = \chi_t(x,t). \tag{3}$$

χ_t is called the relative motion The relative deformation gradient $\mathbf{F}_t(\tau)$ is defined through

$$F_t(\tau) = \frac{\partial \xi}{\partial x}. \tag{4}$$

The history of the stretch tensor $\mathbf{C}_t(\tau)$ is defined through

$$C_t(\tau) = F_t^T(\tau) F_t(\tau). \tag{5}$$

The velocity of the particle $\mathbf{v}(\mathbf{x}, t)$ is defined through

$$v(x,t) = \frac{d\xi}{d\tau}\Big|_{\tau=t}, \tag{6}$$

and the velocity gradient $\mathbf{L}$ is defined

$$L(x,t) = grad\ v(x,t) = \frac{\partial v}{\partial x}. \tag{7}$$

It immediately follows that

$$L(x,t) = \dot{F}F^{-1}. \tag{8}$$

We next record the basic balance laws.

The conservation of mass takes the form

$$\frac{\partial \rho}{\partial t} + div\ (\rho v) = 0, \tag{9}$$

where ρ denotes the density of the fluid. Since we shall be concerned with incompressible materials, we shall require the constraint

$$div\ \ v = 0, \tag{10}$$

or

$$det\ \ F = 1. \tag{11}$$

The balance of linear momentum is given by

$$div\ \ T + \rho b = \rho \frac{dv}{dt}, \tag{12}$$

where **T** denotes the Cauchy stress, **b** the external body force field and d/dt the usual material time derivative.

RESEARCH CARRIED OUT

(i) Flow of fluids of the rate type between two parallel plates rotating with differing angular speeds about a common axis

Amongst the many models for the fluids of the rate type, one that is very popular amongst rheologists, and includes the classical Navier-Stokes model and

the Maxwell model as special cases, is the Oldroyd-B fluid.

The constitutive equation for the extra stress S for the Oldroyd-B fluid is given by (cf. Oldroyd [1])

$$S = p\mathbf{1} + T,$$

$$S + \Lambda_1\left(\dot{S} - LS - SL^T\right) = \mu\left[A_1 + \Lambda_2\left(\dot{A}_1 - LA_1 - A_1L_1^T\right)\right] \tag{13}$$

where the overscore dot denotes the material time derivative, μ is the viscosity, and Λ_1 and Λ_2 are material constants, referred to as the relaxation time and the retardation time, respectively. Also,

$$L = grad\ v, \tag{14}$$

$$A_1 = L + L^T, \tag{15}$$

and

$$\dot{\overline{(\ )}} = \frac{\partial}{\partial t} + [grad\ (\ )]v. \tag{16}$$

We are concerned here with the flow that is induced by the rotation of two infinite disks. The disks are parallel to one another and are located at z = 0 and z = ℓ of a cylindrical polar coordinate system. The disks rotate with angular velocity Ω_1 and Ω_2, respectively. The separation between the disks is d and the fluid velocity v has radial, azimuthal and axial components designated by u, v, and w.

We assume a velocity field of the form (cf. von Karman [2]):

$$\{u, v, w\} = \{rF', rG, -2F\} \tag{17}$$

where F and G are functions of the axial coordinate z only. A lengthy but straight forward manipulation yields the following equations (cf. Ji, Rajagopal & Szeri [3]):

$$S_{rr} + \Lambda_1\left[r\left(F'S_{rr,r} - 2F''S_{zr}\right) - 2F'S_{rr} - 2FS_{rr,z} + GS_{rr,\theta}\right]$$
$$= \mu\left\{2F' - 2\Lambda_2\left[r^2F''^2 - 2\left(F'^2 + FF''\right)\right]\right\}$$

$$S_{r\theta} + \Lambda_1\left[r\left(F'S_{r\theta,r} - 2F''S_{z\theta} - G'S_{zr}\right) - 2F'S_{r\theta} - 2FS_{r\theta,z} + GS_{r\theta,\theta}\right]$$
$$= -2\mu\,\Lambda_2 r^2F''G'$$

$$S_{rz} + \Lambda_1\left[r\left(F'S_{rz,r} - 2F''S_{zz}\right) + F'S_{rz} - 2FS_{rz,z} + GS_{rz,\theta}\right]$$
$$= \mu r\left[F'' + 2\Lambda_2(3F'F'' - FF''')\right]$$

$$S_{\theta\theta} + \Lambda_1\left[r\left(F'S_{\theta\theta,r} - 2G'S_{z}\right) - 2F'S_{\theta\theta} - 2FS_{\theta,z} + GS_{\theta\theta,\theta}\right]$$
$$= \mu\left\{2F' - 2\Lambda_2\left[r^2G'^2 + 2\left(F'^2 + FF''\right)\right]\right\}$$

$$S_{\theta z} + \Lambda_1\left[r\left(F'S_{\theta z,r} - G'S_{zz}\right) + F'S_{\theta z} - 2FS_{\theta z,z} + GS_{\theta z,\theta}\right]$$
$$= \mu r\left[G' + 2\Lambda_2(3F'G' - FG'')\right]$$

$$S_{zz} + \Lambda_1\left\{rF'S_{zz,r} + 4F'S_{zz} - 2FS_{zz,z} + GS_{zz,\theta}\right\}$$
$$= \eta_0\left[-4F' + 8\Lambda_2\left(FF'' - 2F'^2\right)\right]. \qquad (18)$$

The stress components have to satisfy the following set of ordinary differential equations

$$A - 2\Lambda_1(FA' + F'A) = \mu\left[2F' - 4\Lambda_2\left(F'^2 + FF''\right)\right]$$

$$B - 2\Lambda_1(FB' + F'B) = \mu\left[2F' - 4\Lambda_2\left(F'^2 + FF''\right)\right]$$

$$Q - 2\Lambda_1(FQ' + G'P) = -2\mu\Lambda_2 G'^2$$

$$P - \Lambda_1(2FP' - 2F'P + G'R) = \mu\left[G' + 2\Lambda_2(3F'G' - FG'')\right]$$

$$-2\Lambda_1(2FR' - 2F'R) = \mu\left[-4F' + 8\Lambda_2\left(FF' - 2F'^2\right)\right]$$

$$-2\Lambda_1(2FX' - F''Z) = -2\mu\,\Lambda_2 F''^2$$

$$Y - \Lambda_1(2FY' + G'Z + F''P) = -2\mu\,\Lambda_2 F''G'$$

$$Z - \Lambda_1(2FZ' - 2F'Z + F''R) = \mu\left[F'' + 2\Lambda_2(3F'F'' - FF''')\right]. \tag{19}$$

Here A, B, ... etc., are defined through

$$S = \sum_{n=0}^{2} S_n, \tag{20}$$

$$S_0 = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & R \end{pmatrix},$$

$$S_1 = \begin{pmatrix} 0 & 0 & Z \\ 0 & 0 & P \\ Z & P & 0 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} X & Y & 0 \\ Y & Q & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

The balance of linear momentum, in the absence of body force, takes the form

$$\rho r\left(F'^2 - 2FF'' - G^2\right) = -\frac{\partial p}{\partial r} + \frac{\partial S_{rr}}{\partial r} + \frac{\partial S_{rz}}{\partial z} + \frac{S_{rr} - S_{\theta\theta}}{r}$$

$$2\rho r(F'G - FG') = \frac{\partial S_{r\theta}}{\partial r} + \frac{\partial S_{\theta z}}{\partial z} + \frac{2}{r} S_{r\theta}$$

$$4\rho FF' = -\frac{\partial p}{\partial z} + \frac{\partial S_{rz}}{\partial r} + \frac{\partial S_{zz}}{\partial z} + \frac{S_{rz}}{r}. \tag{22}$$

Let

$$p = p_1(z) + \frac{\rho K}{2} r^2 \tag{23}$$

then, substituting for S_{rr}, ..., S_{zz} we obtain the equations of motion as

$$3X - Q + Z' = \rho\left(F'^2 - 2FF'' - G^2 + K\right)$$

$$4Y + p' = 2\rho(F'G - FG')$$

$$p = \frac{\rho K}{2} r^2 + R(z) + 2\int Z(z)\,dz - 2\rho F'^2 + constant$$

$$A \equiv B. \tag{24}$$

To non-dimensionalize the problem we employ the transformation:

$$z = dz, \quad F = \Omega_1 f(z), \quad G = \Omega_1 g(z), \quad E = \nu/\Omega_1 d^2,$$

$$W = \Omega_1 \Lambda_1, \quad \lambda_2 = \Omega_1 \Lambda_2, \quad \beta = \Lambda_2/\Lambda_1, \quad k = K/\Omega_1,$$

$$Q = \frac{\mu\Omega_1}{d^2} Q, \quad P = \frac{\mu\Omega_1}{d} \bar{P}, \quad R = \mu\Omega_1 R,$$

$$X = \frac{\mu\Omega_1}{d^2} X, \quad Y = \frac{\mu\Omega_1}{d^2} Y, \quad Z = \frac{\mu\Omega_1}{d} Z. \tag{25}$$

Here E is the Ekman number, W is the Weissenberg number and β is a measure of retardation time relative to relaxation time.

The above set of ordinary differential equations read, after dropping the overscore bars

$$Q - 2W(fQ' + g'P) = -2\beta W g'^2$$

$$P - W(2fP' - 2f'P + g'R) = g' + 2\beta W(3f'g' - fg'')$$

$$R - 2W(fR' - 2f'R) = -4f' + 8\beta W\left(ff'' - 2f'^2\right)$$

$$X - 2W(f' + f''Z) = -2\beta W f_2''^2$$

$$Y - W(2fY\ + g'Z + f''P) = -2\beta W f''g'$$

$$Z - W(2fZ' - 2f'Z + f''R) = f'' + 2\beta W(3f'f'' - ff''')$$

$$3X - Q + Z' = -\frac{2}{E}\left(ff'' - \frac{f'^2}{2} + \frac{g^2}{2} - \frac{k}{2}\right)$$

$$4Y + P' = \frac{2}{E}(f'g - fg'). \tag{26}$$

We next record the boundary conditions appropriate for the flow under consideration. Since the fluid adheres to the boundary

$$f(0) = 0, \quad f'(0) = 0, \quad g(0) = 1,$$

$$f(1) = 0, \quad f'(1) = 0, \quad g(1) = s \tag{27}$$

where $s \equiv \Omega_2/\Omega_1$. As we have eight differential equations governing the motion of the fluid and the above conditions yield but six of the required twelve boundary conditions, we have to augment (16); to this end we evaluate the stress components Z, P and R at both z = 0 and z = 1, as they are determined in effect by (16). Thus

$$Z(0) = f''(0), \quad Z(1) = f''(1),$$

$$P(0) = g'(0), \quad P(1) = g'(1),$$

$$R(0) = 0, \qquad R(1) = 0. \tag{28}$$

The above system of equations was solved numerically using PITCON on the CRAY Y-MP/48.

The equations exhibit interesting multiple solutions and a detailed discussion of these solutions and plots for the velocities can be founded in the attached paper titled "Multiplicity of solutions in von Karman flows of viscoelastic fluids".

(ii) <u>Flow of non-Newtonian fluids of the integral type between parallel plates rotating about distinct axes</u>

While rate type and differential type models are useful in describing dilute polymeric solutions and materials with fading memory, when it comes to describing the behavior of polymeric material which have finite memory we have to use integral representations for the stress which incorporates the history of the deformation in the model. The K-BKZ model (cf. Kaye [4], Bernstein, Kearsley & Zapas [5]) has proved to be quite successful in describing behavior of a wide class of polymeric fluids and in addition the model has been shown to have a firm footing from the point of view of statistical theories for modeling.

Here, we shall describe the flow of a K-BKZ fluid in an orthogonal rheometer (cf. Maxwell & Chartoff [6], Rajagopal [7]).

A detailed discussion of the above equations for a special subclass of K-BKZ fluid, the Wagner fluid, can be found in the attached paper titled "Flow of K-BKZ fluids between parallel plates rotating about distinct axes: shear-thinning and inertial effects" while the flow of another popular subclass, the Curie fluid can be

found in the attached paper titled "Flow of viscoelastic fluids between rotating plates about distinct axes".

The Cauchy stress **T** in the K-BKZ fluid has the structure

$$T = -p\mathbf{1} + 2\int_{-\infty}^{t} \left\{U_1 C_t^{-1}(\tau) - U_2 C_t(\tau)\right\} dt, \tag{29}$$

where

$$C_t(\tau) = F_t^T(\tau) F_t(\tau). \tag{30}$$

In (29) U denotes the strain energy function for the viscoelastic fluid and is a function of the principal invariants of $\mathbf{C}_t(\tau)$ and $\mathbf{C}_t^{-1}(\tau)$:

$$U = U(I_1, I_2, t-\tau), \tag{31}$$

$$I_1 = tr\, C_t^{-1}\tau, \quad I_2 = tr\, C_t \tau, \tag{32}$$

and

$$U_i = \frac{\partial U}{\partial I_i}, \quad i = 1,2. \tag{33}$$

Let $\mathbf{x} = (x, y, z)$ denote the position occupied by the same particle **X** at time t. It follows that

$$\dot{\xi} = -\Omega(\eta - g(\varsigma)), \tag{34}$$

$$\dot{\eta} = -\Omega(\xi - f(\varsigma)), \tag{35}$$

$$\dot{\varsigma} = 0, \tag{36}$$

with

$$\xi(t) = x, \quad \eta(t) = y, \; and \quad \varsigma(t) = z.$$

Rajagopal [7] has shown that the motion (34)-(36) is a motion with constant principal relative stretch history. In such motion, the stress is determined by the first three Rivlin-Ericksen tensors $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$.

However, for the motion under consideration

$$A_3 = -\Omega^2 A_1 . \tag{37}$$

Thus, the stress is given by

$$T = -p\mathbf{1} + \hat{f}(A_1, A_2) . \tag{38}$$

The balance of linear momentum has the form:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\partial \varphi}{\partial x} + \Omega^2[x - f] + \frac{1}{\rho} h_1(f', g', f'', g''), \tag{39}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -\frac{\partial \varphi}{\partial x} + \Omega^2[y - f] + \frac{1}{\rho} h_2(f', g', f'', g''), \tag{40}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -\frac{\partial \varphi}{\partial x} + \frac{1}{\rho} h_3(f', g', f'', g'') . \tag{41}$$

The specific constitutive equations determine the functions $\mathbf{h}_1$, $\mathbf{h}_2$, and $\mathbf{h}_3$. This can then be substituted into (39)-(41) and the appropriate partial differential equations analyzed. Notice that (39)-(41) are of second order and hence the no-slip boundary conditions are sufficient for determinacy. The appropriate boundary conditions for the velocity field are (cf. Figure 2)

$$u = \frac{\Omega a}{2} - \Omega y, \quad v = \Omega x, \quad w = 0 \quad at \quad z = h, \tag{42}$$

$$u = \frac{\Omega a}{2} - \Omega y, \quad v = \Omega x, \quad w = 0 \quad at \quad z = 0, \tag{43}$$

and

$$u \to \mp\infty, \quad v \to \pm\infty, \quad as \quad x, y \to \pm\infty. \tag{44}$$

It follows from (42), (43), and (34)-(36) that

$$f(h) = f(0) = 0, \tag{45}$$

$$g(h) = \frac{a}{2}, \quad g(0) = -\frac{a}{2}. \tag{46}$$

In eliminating the pressure field we have raised the order of the equations. Thus the boundary conditions (45) and (46) are not sufficient to determine the solution to the system (39) and (41). We augment the number of boundary conditions by recognizing that the locus of the centers of rotation cuts the plane
$z = 0$ at some point, say (ℓ_1, ℓ_2). Thus

$$f\left(\frac{h}{2}\right) = \ell_1, \quad g\left(\frac{h}{2}\right) = \ell_2. \tag{47}$$

However, if we restrict ourselves to solutions which have midplane symmetry, then

$$f\left(\frac{h}{2}\right) = 0, \quad g\left(\frac{h}{2}\right) = 0. \tag{48}$$

For the rest of this section we restrict ourselves to a discussion of solutions which possess midplane symmetry. For the motion under consideration, a lengthy but straightforward computation yields

$$C_t(\tau) = 1 - \frac{s}{\Omega} A_1 + \frac{(1-c)}{\Omega^2} A_2 \tag{49}$$

and

$$C_t^{-1}(\tau)=1+\frac{s}{\Omega}\left[1+2(1-c)(f'^2+g^2)\right]A_1-\frac{(1-c)}{\Omega^2}\left[1+2(1-c)(f'^2+g^2)\right]A_2+\frac{s^2}{\Omega^2}A_1^2$$

$$+\frac{(1-c)}{\Omega^4}A_2^2+\frac{s(1-c)}{\Omega^3}(A_1A_2+A_2A_1), \tag{50}$$

where

$$s\equiv sin\Omega(t-\tau),\quad c\equiv cos\Omega(t-\tau)\,. \tag{51}$$

Also, notice that

$$I_1(t,\tau)=I_2(t,\tau)=3+2(1-c)(f'^2+g'^2)\equiv I(\Omega(t-\tau),z)\,. \tag{52}$$

It follows from (39)-(41), (40), and (50) that

$$\frac{d}{dz}\{f'B(\kappa)+g'A(\kappa)\}=\rho\,\Omega^2 f, \tag{53}$$

$$\frac{d}{dz}\{-f'A(\kappa)+g'B(\kappa)\}=\rho\,\Omega^2 g, \tag{54}$$

where

$$\kappa\equiv\left(f'^2+g'^2\right)^{1/2} \tag{55}$$

and

$$A(\kappa)=2\int_0^\infty \tilde{U}\left[3+2(1-cos\,\Omega\alpha)\kappa^2,\alpha\right]sin\Omega\alpha\,d\alpha, \tag{56}$$

$$B(\kappa)=2\int_0^\infty \tilde{U}\left[3+2(1-cos\,\Omega\alpha)\kappa^2,\alpha\right](1-cos\,\Omega\alpha)d\alpha, \tag{57}$$

$$\tilde{U}(I,\alpha)\equiv U_1(I,I,\alpha)+U_2(I,I,\alpha)\,. \tag{58}$$

Let t_x and t_y denote the x and y components of the traction on the plates. It follows that

$$t_x(h) = +B(\kappa,\Omega)f'(h) + A(\kappa,\Omega)g'(h), \tag{59}$$

$$t_y(h) = -A(\kappa,\Omega)f'(h) + A(\kappa,\Omega)g'(h)\ . \tag{60}$$

Thus the material parameters A(κ, Ω) and B(κ, Ω) can be expressed in terms of t_x and t_y as (cf. Bower, Wineman & Rajagopal [8]):

$$A(\kappa,\Omega) = \frac{-1}{\kappa^2}\left[t_y f' + t_x g'\right], \tag{61}$$

$$B(\kappa,\Omega) = \frac{1}{\kappa^2}\left[t_x f' + t_y g'\right]. \tag{62}$$

Dai, Rajagopal and Szeri [9] studied the flow of a Currie fluid for which the strain energy U is given by

$$U = U(I_1, I_2, s) = -\dot{G}(s)\left[5\,\ell n(J-1) - 9.73\right], \tag{63}$$

where

$$J = I_1 + 2(I_2 + 3.25)^{1/2} \tag{64}$$

and

$$\dot{G}(s) = -Ce^{-\lambda s}\ . \tag{65}$$

This model is a very popular model for polymeric materials.

In a recent study Zhang and Goddard [10] studied the flow of a Currie fluid in an orthogonal rheometer and found problems with convergence when values of certain parameters were large. However, Dai, Rajagopal and Szeri [9] were able to go well past the of the parameters for which Zhang and Goddard [10] had difficulty, using an analytic continuation technique. The details of the results can be found in the paper of Dai, Rajagopal and Szeri [9] appended to this report. Of particular interest is the fact that Dai, Rajagopal and Szeri [9] were able to study

the problem for Reynolds numbers as high as 10,000 and found the presence of interesting boundary layers.

(iii) Flow of non-Newtonian fluids due to torsional and longitudinal oscillations of a cylinder

There are many important applications like the drilling for oil where we encounter the flow induced due to the torsional and longitudinal oscillations of a cylinder in a non-Newtonian fluid. Here, we discuss the flow in the case of a general incompressible simple fluid (cf. Truesdell & Noll [11]). The Cauchy stress **T** in a homogeneous incompressible simple fluid is given by

$$T = -p\mathbf{1} + \mathcal{F}_{s=0}^{\infty}\left[F_t(t-s)\right], \tag{66}$$

where $\mathcal{F}_{s=0}^{\infty}$ is a general functional and $\mathbf{F}_t\,(t-s)$ is the relative deformation gradient. Coleman and Noll [12] developed a method for approximating (66) within the context of retarded motions. They showed that up to order four

$$T = -p\mathbf{1} + \sum_{i=1}^{4} S_i \tag{67}$$

$$S_1 = \mu A_1, \tag{68}$$

$$S_2 = \alpha_1 A_2 + \alpha_2 A_1^2, \tag{69}$$

$$S_3 = \beta_1 A_3 + \beta_2 (A_1 A_2 + A_2 A_1) + \beta_3 (tr A_1^2) A_1, \tag{70}$$

$$S_4 = \gamma_1 A_4 + \gamma_2 (A_1 A_3 + A_3 A_1) + \gamma_3 A_2^2 + r_4 [A_2 A_1^2 + A_1^2 A_2] + \gamma_5 (tr A_1^2) A_2 + \gamma_6 (tr A_1^2) A_1^2 + \gamma_7 (tr A_3) A_1 + \gamma_8 (tr A_2 A_1) A_1, \tag{71}$$

where $\mathbf{A}_1$ is as defined earlier and (cf. Rivlin & Ericksen [13]):

$$A_n = \frac{d}{dt} A_{n-1} + A_{n-1}(grad\ v) + (grad\ v)^T A_1 . \tag{72}$$

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Using the above approximation for the stress, we studied the longitudinal and torsional oscillations of a solid cylinder in a simple fluid. We assume a velocity field of the form (cf. Rajagopal, Kaloni & Tao [14])

$$v_r = 0, \quad v_\theta = v(r,t), \quad v_z = w(r,t) \tag{77}$$

where v_r, v_θ and v_z denote the components of the velocity in the r, θ and z directions. We assume that the velocity components v and w and the pressure p can be expanded as a power series in Ω, for sufficiently small Ω (Ω being the frequency of torsional oscillations):

$$v = \sum_{n=1} \Omega^n v^{(n)} + 0\left(\Omega^{n+1}\right), \tag{78}$$

$$w = \sum_{n=1} \Omega^n w^{(n)} + 0\left(\Omega^{n+1}\right), \tag{79}$$

$$p = \sum_{n=1} \Omega^n p^{(n)} + 0\left(\Omega^{n+1}\right). \tag{80}$$

Substituting the above expansions into the balance of linear momentum and equating the coefficients of Ω^n, n = 1, ... leads to a hierarchy of equations which can be solved successively. Here, we shall not document these equations as they are exceedingly cumbersome. We refer the reader to the attached paper titled "Longitudinal and torsional oscillations of a solid cylinder in a simple fluid" for details of the calculations.

The appropriate boundary conditions are

$$v = \Omega v\, cos\, \Omega t\, \boldsymbol{e}_\theta + \Omega w\, cos\, \lambda \Omega t\, \boldsymbol{e}_z, \tag{81}$$

since we are assuming that the frequency of oscillations in the circumferential and longitudinal directions are not necessarily the same.

Such a perturbation approach is very popular and has been used by many in the field of non-Newtonian fluid mechanics. However, such an approach is not without drawbacks and these are also discussed in some detail in the above mentioned paper.

The study delineates the effects of the various material parameters on the wall shear stress. In general, in keeping with expectations, an increase in the higher order viscosities like β_3 leads to an increase in the wall shear stress, as also does an increase in Ω. The effect of variation in the normal stress moduli α_1 is quite dramatic. Details regarding the variations of the shear stress $T_{r\theta}$, v and w with the various material parameters are provided in Figures 1-12 of the paper cited above.

(iv) Flow of non-Newtonian fluids in pipes of varying cross-sections

The flow of non-Newtonian fluids through axially symmetric pipes of varying cross-sections has relevance to several technologically significant problems in biofluid dynamics and extrusion of polymeric materials. This problem has been studied within the context of the classical linearly viscous fluid model, and also specific non-Newtonian fluid models (mostly power law models) by several authors.

We studied the flow of a four constant Oldroyd fluid (cf. Oldroyd [15]) in an axi-symmetric pipe using a perturbation approach, using two parameters: ε which describes the departure of the cross-section of the pipe from circularity and W (Weissenberg number) which is the ratio of the relaxation time to a characteristic time scale for the problem.

The pressure drop along the axis of the tube and the shear stress are calculated up to second order. The manner in which the elasticity of the fluid alters the flow pattern, the effect of shear-thinning on the flow, and the formation of eddies and the manner in which the flow separates are all studied in detail. When appropriate parameters are set to zero, the results reduce to the results established for the Navier-Stokes fluid. We find that elastic effects can initiate separation at much lower Reynolds number than for a Newtonian fluid. Moreover, the wall shear stress can change dramatically with changes in some of the non-Newtonian parameters. For instance, increasing the retardation parameter decreases the wall shear stress, while increasing another non-Newtonian parameter (α) which appears in the model can significantly increase the wall shear stress.

The Cauchy stress **T** in a four constant Oldroyd fluid is related to the fluid motion in the following manner (cf. Oldroyd [ ]):

$$\bar{T} + \lambda_1\left\{\frac{D\bar{T}}{Dt} - \frac{\alpha}{2}\left(\bar{A}\bar{T} + \bar{T}\bar{A}\right)\right\} = 2\eta_0\lambda_2\left\{\frac{D\bar{A}}{DT} - \frac{\alpha}{2}\left(A^2\right)\right\} \tag{82}$$

where D/Dt represents the co-rotational or Jaumann time derivative, given by (for any symmetric second order tensor **B**)

$$\frac{DB}{DT} = \frac{\partial B}{\partial t} = [grad\ B]u + WB - BW, \tag{83}$$

and

$$\bar{A} = \nabla u = (\nabla\bar{u})^T, \quad 2\bar{W} = \nabla u - (\nabla\bar{u})^T. \tag{84}$$

Also, η_0 is the viscosity of the fluid at the zero shear rate, λ_1, is the relaxation

time, λ_2 is the retardation time and α is a parameter whose value lies between 0 and 1. When $\alpha = 0$, the above model is equivalent to Jeffery's version of the Oldroyd model whereas when $\alpha = 1$ it reduces to the limiting form of Walter's four constant version of Oldroyd's model. In viscoelastic materials it seems reasonable to assume that $0 < \alpha < 1$. In this range predicts shear rate dependent viscosity, normal stress effects in shear flow and a reasonable behavior in elongatial flow.

We assume the flow to be axisymmetric so that the components of the velocity ($\bar{u}$, $\bar{v}$, $\bar{w}$) in (x, r, θ) direction satisfy the constraint of incompressibility:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial r} + \frac{\bar{v}}{r} = 0. \tag{85}$$

We introduce the axi-symmetric stream function ψ through

$$\bar{u} = \frac{\Psi_r}{r} \quad and \quad \bar{v} = -\frac{\Psi_r}{r}, \tag{86}$$

and the vorticity component Ω through

$$\Omega = \bar{u}_r - \bar{v}_x = \frac{\bar{\Psi}_{xx}}{r} + \left(\frac{\bar{\Psi}_r}{r}\right)_r, \tag{87}$$

where the subscript denotes partial differention with respect to that variable.

The appropriate boundary condition on ∂D are

$$\bar{u} + \left(\frac{da}{dx}\right)\bar{v} = 0 \quad on \quad r = a(x)$$

$$\psi = 0, \quad \bar{v} = 0, \quad and \quad \bar{u}_r = 0 \quad on \quad r = 0. \tag{88}$$

Equations $(88)_{1,2}$ are a consequence of the adherence boundary condition and imply that there is no fluid motion either tangential or normal to the wall. The conditions $(88)_{3,4}$ are essentially symmetry conditions on the axis of the tube. The constant flow rate taking place through any cross-section of the tube is given by

$$\int_0^a\int_0^{2\pi} r\bar{u}\,d\theta\,dr = 2\pi Q = constant\,.$$

The boundary conditions $(88)_{1\text{-}4}$ can be expressed in terms of $\overline{\Psi}$ by entering (88) into (86):

$$\overline{\Psi}_r = 0,\quad \overline{\Psi} = \Psi_0 \quad on \quad r = a(x)$$

$$\overline{\Psi} = 0 \quad on \quad r = 0$$

$$\frac{\overline{\Psi}_x}{r} \to 0,\quad \left(\frac{\overline{\Psi}_r}{r}\right)_r \to 0 \quad as \quad r \to 0\,.$$

We now proceed to non-dimensionalize the governing equations. Let

$$\bar{u} = \frac{\Psi_0}{a_0^2}u,\quad \bar{v} = \frac{\Psi_0}{a_0^2}v,\quad \bar{T} = \frac{\eta_0\Psi_0}{a_0^3}\bar{T},\quad A = \frac{\Psi_0}{a_0^3}A,\quad W = \frac{\Psi_0}{a_0^3}W$$

$$p = \left(\eta_0\Psi_0/a_0^3\right)q,\quad \overline{\Psi} = \Psi_0\phi(z,x^*,\varepsilon),\quad \Omega = \left(\Psi_0/a_0^3\right)\omega(z,x^*,\varepsilon),$$

$$a(x,\varepsilon) = a_0 s\left(\varepsilon x/a_0\right),\quad (0<\varepsilon<<1),\quad x^* = \varepsilon x/a_0,\quad z = r/a_0, \tag{91}$$

where p is the pressure, and a_0 is a constant characteristic radius of the tube. The function $s(\varepsilon x/a_0)$ is such that in the limit $\varepsilon \to 0$, the tube is of constant radius, and the variation of a with respect to the axial coordinate x depends upon εx rather than x alone. Here z is the normalized radial coordinate and x^* is a

"slowly varying" normalized axial coordinate. On substituting (91) into (82) we obtain the dimensionless form of the constitutive equation as

$$T = 2A + 2\varepsilon_1 N_i \left\{ \frac{DA}{Dt} = WA - AW - 2\alpha\left(A^2\right) \right\}$$

$$-Wi\left\{ \frac{DT}{Dt} + WT - TW - \alpha(AT + TA) \right\}, \tag{92}$$

where

$$Wi = \frac{\lambda_1 \Psi_0}{a_0^3} \ (Weissenberg\ number)$$

and

$$\varepsilon_1 = \frac{\lambda_2}{\lambda_1}.$$

The dimensionless form of the equation of motion in the absence of body forces is given by

$$div\ T - grad\ p = Re \frac{dv}{dt}, \tag{93}$$

where $Re = \Psi_0 / a_0 \nu$ is a Reynolds number.

On substituting (91) into (90) we get the dimensionless boundary conditions as

$$\left.\begin{array}{l} \phi_z = 0 \\ \phi = 1 \end{array}\right. \quad on\ z = s(x^*),$$

$$\phi = 0 \ \ on \ \ z = 0, \ \ \phi_{zz}/z \to 0 \ \ as \ \ z \to 0, \ \ (\phi_z/z)_z \to 0 \ \ as \ \ z \to 0. \tag{94}$$

We solve the system of equations (92)-(94) for small Weissenberg and Reynolds number. The previous low Reynolds number solutions for the Navier-Stokes

equations can be readily retrieved from our solution and our results reduce to the results of Manton [16] when the Weissenberg number Wi = 0.

First, assuming that Re is fixed and is of order unity, a solution is obtained in the form of a power series in Wi for the quantities ϕ, ω, u, v, q, **T**, **e**, and **w** (cf. Kasivishvanathan, Kaloni & Rajagopal [17]):

$$A = A_0 + Wi\,A_1 + (Wi)^2 A_2 + \ldots \tag{95}$$

where A in turn stands for ϕ, ω, **u**, **v**, q, **T**, **e** and **w**.

Details of the results can be found in the enclosed paper titled "Flow of a non-Newtonian fluid through axi-symmetric pipes of varying cross-sections".

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